HAMLET AND PFISTER FORMS*
(A TRAGEDY IN FOUR ACTS)

Ján Mináč

Abstract. In the mid-1960s A. Pfister discovered extraordinary, strongly multiplicative forms which are now called Pfister forms. From that time on, these forms played a dominant role in the theory of quadratic forms. One of the key properties of a Pfister form q is that q extended to a suitable transcendental extension, has the polynomial q as its similarity factor. Pfister's original proof used clever matrix calculations. Here we show that the desired isometry is induced by the multiplication of a suitable field element.

We further consider the surprising possibility that Pfister's forms were already known by Hamlet, Rosencrantz and Guildenstern, and that they in fact led to a terrible tragedy which is yet filled with a haunting beauty and mystery that can still inspire us to this day.

THE CAST OF CHARACTERS:

CLAUDIUS .... Ján Mináč

HAMLET .... Ján Mináč

THE QUEEN .... Leslie Hallock

**Performed at Oberwolfach in May, 1992.*

ACT I **Hamlet's Terrible Revenge - The Question** (TRUMPET sounds)

THE QUEEN Something is rotten in the state of Denmark. How weary, stale, flat and unprofitable seem to me all uses of this world!

(walking, throwing his hands up in despair)

HAMLET Evil is here, I swear it! In the midst of us. I will avenge it! I will kill neither you nor the King, my dear mother. That would be too easy for you! I will not let you sleep, eat, nor rest. I will torture you with the mystery of Pfister forms!

THE QUEEN (falling on her knees, cries out)

Oh! Have mercy, Hamlet! Please don't!

HAMLET Frailty, thy name is woman! Only a solution of this great puzzle of quadratic forms will free you from my spell!

THE QUEEN Forms!? Did you say quadratic forms!? Hamlet, don't we have enough forms!? We already have tax forms, insurance forms, government forms, claim forms, request forms; even execution forms - I am not filling out anything! If you want some form in any case, then have a uniform! You can be a soldier!

HAMLET No, mother. You will not get rid of me so easily! Let us waste no time! Let us share the wisdom of my prophet Horatio.

Assume that F is a field with $1+1 \neq 0$. $K = F(X_1, \ldots, X_{2n})$, φ is an anisotropic quadratic form over F, $\varphi = \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$. φ_K is the extended form of F to K.

As my prophet told me, somewhere in Germany a few centuries later, a man named Pfister will discover that $\varphi_K \cong \varphi(X_1, \ldots, X_{2^n})\varphi_K$.

THE QUEEN Oh! I am lost! I am utterly lost! Words! Words! Words! I don't understand them.

HAMLET You! You who married two days after my father's burial! You who married the brother of your husband. You who doesn't have any decency! YOU LISTEN!

THE QUEEN You are mad! Oh, my lord! (afraid)

HAMLET Pfister will publish a very beautiful matrix proof, and less than 2 years later, Witt with his usual wit and magic, will utter a magic word, "Runde", and the whole proof will collapse into a few lines. It will be a joke.

THE QUEEN Brevity is the soul of Witt!

HAMLET And also of Pfister. They always produce elegant and short proofs. But nevertheless, for some time there will be no proof which uses some underlying algebra with its multiplication and norm-like map as can be done in the case of $2-, 4-, 8-$, and 16 dimensional Pfister forms using quadratic extensions, quaternion and Cayley-Dickson algebras respectively.

THE QUEEN But our good Horatio tells me that Shapiro will make some progress.

HAMLET Yes. He will. He will succeed in a quite intriguing construction of Pfister forms using the multiplication in Clifford algebras. However, for some time it will not be clear how to use his results to obtain multiplicative properties.

AND THIS IS YOU AND YOUR HUSBAND'S TASK, MY DEAR MOTHER! YOU WILL BE BEWITCHED, RESTLESS, WORRIED ... UNTIL YOU FIND THIS PROOF! THIS IS MY REVENGE!

TRUMPET FLASH CARD

ACT II **A Quest for an Algebra and a Marriage in Danger**

THE QUEEN Claudius, you are a king; you have power, money, gold; everything! You must find an answer! What would be the right generalization of a quadratic field, a quaternion algebra, or a Cayley-Dickson algebra?

THE KING (sadly) My dear Queen, unfortunately there is no royal road to mathematics. Horatio even told me that Hurwitz and other mathematicians will prove that no 16 dimensional algebra will work. We are lost.

THE QUEEN I should have never married you. We should have never made a union. We cannot even prove a theorem together!

THE KING We shall, we will, we must!

Look, a 1-fold Pfister form $\langle 1, a\rangle$ is a norm form from the single quadratic extension $L = F(\sqrt{-a})$, so if there is any justice in the world, then an n-fold Pfister form must be the norm from the multiquadratic extension $L = F(\sqrt{-a_1}, \ldots, \sqrt{-a_n})$.

THE QUEEN By my lord, don't you know that there is no justice in the world? It will not work!

THE KING (whispers) I found the following interesting scrap of paper in Hamlet's chamber. Poor Hamlet, he doesn't know that I have all of the keys to this castle (still looking at various scraps of paper). A love letter to Ophelia - the idiot! (He throws the letter away.)

Look, look! Here is something of interest.

$$L = K\left(\sqrt{-a_1}, \sqrt{-a_2\frac{\widehat{\psi}_1}{\psi_1}}, \cdots, \sqrt{-a_n\frac{\widehat{\psi}_{n-1}}{\psi_{n-1}}}\right) \text{ where}$$

$\psi_1 = X_1^2 + a_1X_2^2, \ \widehat{\psi}_1 = X_3^2 + a_1X_4^2,$

$\psi_2 = X_1^2 + a_1X_2^2 + a_2X_3^2 + a_1a_2X_4^2, \ \widehat{\psi}_2 = X_5^2 + a_1X_6^2 + a_2X_7^2 + a_1a_2X_8^2$, etc.

Observe that $L \cong \bigotimes_{i=1}^{n} L_i$ as K-vector spaces, where $L_1 = K(\sqrt{-a_1})$,

$L_i = K\left(\sqrt{-a_i\frac{\widehat{\psi}_{i-1}}{\psi_{i-1}}}\right)$ for $i \geq 2$. $\ell_1 \otimes \cdots \otimes \ell_n \to \ell_1 \cdots \ell_n$ as any fool will tell you. This must be our algebra!

THE QUEEN You are a thief - a scoundrel! How could you do it? How could you steal!?

THE KING Thus conscience does make cowards of us all. But I have no conscience; so I am not a coward; therefore I steal. Alright, look!

$L = K(\sqrt{-a}),\ \omega = N_{L/K}$-1-fold Pfister $\qquad L = K\left(\sqrt{-a_1}, \sqrt{-a_2\frac{\widehat{\psi}_1}{\psi_1}}, \cdots, \sqrt{-a_n\frac{\widehat{\psi}_{n-1}}{\psi_{n-1}}}\right)$

$\Theta = X_1 + X_2\sqrt{-a} \qquad \omega = ? \qquad \Theta = ?$

$\omega(Z_1 + Z_2\sqrt{-a})$ is

$Z_1^2 + Z_2^2 a = N_{L/K}(Z_1 + Z_2\sqrt{-a}) \qquad \omega(\Theta \cdot \ell) = \psi(X_1, \cdots, X_{2^n})\omega(\ell)$

$$\begin{aligned}\omega(\Theta\ell) &= \omega(\Theta)\omega(\ell)\\ &= \psi(X_1, X_2)\omega(\ell).\end{aligned}$$

ψ is just φ evaluated at $X_1, \cdots, X_{2^n}, \psi \in K$.

We just have to find a good way of describing ω on L and also a good element $\Theta \in L$.

THE QUEEN You are naive, my lord. This cannot work. I am sure that you need a noncommutative nonassociative algebra; something horrible, something beyond our wildest imaginings. You need more matter with less art.

THE KING How dare you call me naive! I steal, I murder, I have power – I am the king! I cannot be naive. I order Θ to exist!

THE QUEEN (ironically) Well, if you are so sure, find it!

KING (laughing) I have it! (ha,ha) Polonius stole it from Horatio's room. Here it is:

$$\Theta = (X_1 + X_2\sqrt{-a_1})\left(1 + \sqrt{-a_2\frac{\widehat{\psi}_1}{\psi_1}}\right)\cdots\left(1 + \sqrt{-a_n\frac{\widehat{\psi}_{n-1}}{\psi_{n-1}}}\right).$$

THE QUEEN I hate you passionately!

TRUMPET FLASH CARD

ACT III **The Proof**

THE QUEEN (walking) To prove or not to prove, that is the question.

CLAUDIUS To prove! To prove! But what to prove?

THE QUEEN You proved that you are a scoundrel and a common thief! That is all that you have achieved.

CLAUDIUS I know all of this. But I still love you, still adore you. I need your help. We have to show that

$$\omega(\Theta \cdot \ell) = \psi(X_1, \ldots, X_{2^n})\omega(\ell).$$

Rosencrantz and Guildenstern found out how Θ should be defined. $\omega = \bigotimes_{i=1}^{n} \omega_i$, where ω_i is the norm form $N_{L_i/K}$ on L_i. You remember that $L \cong \bigotimes_{i=1}^{n} L_i$ as K-vector spaces.

THE QUEEN Poor Rosencrantz and Guildenstern! I heard that they were executed in England.

CLAUDIUS There is a divinity that shapes our ends, rough-hew them how we will. In any case, before their death they managed to tell me the proof. Look! It works beautifully:

Define B to be a symmetric bilinear form associated with ω and define $\widetilde{\omega}$ by the equation $\widetilde{\omega}(\ell) = \omega(\Theta\ell)$. Let B be also the symmetric bilinear form associated with ω. Then we have for all $h_1, \ldots, h_n$ and $\ell_1, \ldots, \ell_n \in L$

$$\begin{aligned}
\widetilde{B}(h_1 \otimes \cdots \otimes h_n, \ell_1 \otimes \cdots \otimes \ell_n) &= \widetilde{B}(h_1 \cdots h_n, \ell_1 \cdots \ell_n) \\
&= B(\Theta h_1 \cdots h_n, \Theta \ell_1 \cdots \ell_n) \\
&= B\left(\left(\prod_{i=1}^{n} \Theta_i h_i\right), \prod_{i=1}^{n} (\Theta_i \ell_i)\right) \\
&= \prod_{i=1}^{n} B_i(\Theta_i h_i, \Theta_i \ell_i) \\
&= \prod_{i=1}^{n} N_{L_i/K}(\Theta_i) \prod_{i=1}^{n} B_i(h_i, \ell_i) \\
&= \psi(X_1, \ldots, X_{2^n}) B(h_1 \otimes \cdots \otimes h_n, \ell_1 \otimes \cdots \otimes \ell_n).
\end{aligned}$$

Observe

$$\prod_{i=1}^{n} N_{L_i/K}(\Theta_i) = \psi(X_1, \ldots, X_{2^n}).$$

Indeed $N_{L_1/K}(\Theta_1) = X_1^2 + a_1 X_2^2 = \psi_1(X_1, X_2)$.

$$N_{L_1/K}(\Theta_1) N_{L_2/K}(\Theta_2) = (X_1^2 + a_1 X_2^2)\left(1 + a_2 \frac{x_3^2 + a_1 X_4^2}{X_1^2 + a_1 X_2^2}\right) = \psi_1(X_1, X_2, X_3, X_4)$$

etc.

THE KING Finally from the linearity reason we conclude

$$\widetilde{B} = \psi(X_1, \dots, X_{2^n})B.$$

THE QUEEN What a pity Rosencrantz and Guildenstern were executed. They were gifted!

THE KING Well, this is the reason they were executed. No decent monarchy can support creative people.

TRUMPET FLASH CARD

ACT IV **Pfister Matrices and the Death of a King**

THE QUEEN So we found that a suitable multiplication in some field does the trick. What is it good for?

THE KING I guess, if you have gained such good insight as we have obtained, you can try to generalize; for example, why not replace our field extensions L_i/K by suitable cubic extensions; perhaps one can discover cubic Pfister forms, or find nice subspaces of L/K, or study the spaces of similarities; perhaps one can make further progress with Shapiro's conjecture. In particular, perhaps one can even solve his "Pfister-factor" conjecture.

THE QUEEN Why would one like to solve a conjecture?

THE KING If it will be solved, it will be discovered now; if it won't be discovered, it won't be now; if it isn't now, yet it will be solved; readiness is all.

THE QUEEN What will happen if we can solve it?

THE KING The cat will mew, and the dog will have his day.

THE QUEEN I wonder whether poor Rosencrantz and Guildenstern's proof has something in common with Pfister and Witt's proofs.

THE KING Yes. They told me that their proof essentially gives a basis-free variation of Pfister's proof; it shows that Pfister matrices correspond to field multiplication. When compared with Witt's proof, Rosencrantz and Guildenstern's proof not only shows that ψ_K and $\psi(X_1, \ldots, X_{2^n})\varphi_K$ are isometric, but also constructs isometry.

THE QUEEN Rosencrantz and Guildenstern formed such a cute isometric pair! One did exactly the same as the other. They were indistinguishable. No wonder they found a good isometry!

(Queen sips from wine cup)

THE KING Don't drink that wine! That wine is for Hamlet.

THE QUEEN O.K., O.K. I thought that there was enough wine in our kingdom.

THE KING Not any more! We have to be careful. The economy is very bad. We have to be as stingy as possible.

In any case, we can pick a good orthogonal ordered basis B of our field L over K and identify

$$L \longleftrightarrow \qquad \left\{ \begin{pmatrix} Y_1 \\ \vdots \\ Y_{2^n} \end{pmatrix}, Y_i \in K \right\} = V$$

$$e = \Sigma Y_i b_i \qquad \begin{pmatrix} Y_1 \\ \vdots \\ Y_{2^n} \end{pmatrix}$$

$$\omega \qquad\qquad \varphi$$

$$\widetilde{\Theta}\colon L \to L \qquad \widetilde{\widetilde{\Theta}}\colon V \to V$$

$$e \to \Theta\ell \qquad \begin{pmatrix} Y_1 \\ \vdots \\ Y_{2^n} \end{pmatrix} \to T \begin{pmatrix} Y_1 \\ \vdots \\ Y_{2^n} \end{pmatrix}$$

where T is a matrix of $\tilde{\tilde{\Theta}}$ written with respect to B

$$\omega(\Theta\ell) = \omega(\Theta)\omega(\ell) \qquad \varphi\left(T\begin{pmatrix} Y_1 \\ \vdots \\ Y_{2^n}\end{pmatrix}\right) = \varphi(X_1,\ldots,X_{2^n})\varphi(Y_1,\ldots,Y_{2^n})$$

which shows beautifully that φ is strictly multiplicative in the Pfister sense.

(The Queen tries to secretly drink the same wine as before. The King observes it. He rushes to the queen, takes the wine out of her hand. He is furious.)

THE KING I TOLD YOU NOT TO DRINK THAT WINE!! We can't afford it! Would you like to ruin our whole kingdom? You will become an alcoholic.

THE QUEEN You horrible man! (to audience) What a cheap man my husband is! All that I wanted was to sweeten my tongue! (cynically) Lend me this wine, my Lord. I will return it to you.

THE KING Neither a borrower nor a lender be. NO!

Listen my dear. Don't drink it. Rather, pay attention to this beautiful example:

$\varphi = \langle 1, a_1\rangle \otimes \langle 1, a_2\rangle$. Then $K = F(X_1, X_2, X_3, X_4)$.

$$L = F\left(\sqrt{-a_1}, \sqrt{-a_2\frac{X_3^2 + a_1X_4^2}{X_1^2 + a_1X_2^2}}\right) \qquad T = \begin{pmatrix} X_1 & -aX_2 & S & \\ X_2 & X_1 & & \\ X_3 & -a_1X_4 & X_1 & -a_1X_2 \\ X_4 & X_3 & X_2 & X_1 \end{pmatrix}$$

$$B = \left\{1, \sqrt{-a_1}, \sqrt{-a_2\frac{\hat{\psi}_1}{\psi_1}}\,\frac{X_1 + X_2\sqrt{-a_1}}{X_3 + X_4\sqrt{-a_1}}, \sqrt{-a_1}\sqrt{-a_2\frac{\hat{\psi}_1}{\psi_1}}\,\frac{X_1 + X_2\sqrt{-a_1}}{X_3 + X_4\sqrt{-a_1}}\right\}$$

where

$$S = \frac{-a_2}{X_1^2 + a_1X_2^2}\left(\begin{array}{c|c} X_1^2X_3 + 2a_1X_1X_2X_4 - a_1X_2^2X_3 & -2a_1X_1X_2X_3 - a_1^2X_2^2X_4 + a_1X_1^2X_4 \\ \hline 2X_1X_2X_3 - X_1^2X_4 + a_1X_2^2X_4 & X_1^2X_3 - a_1X_2^2X_3 + 2a_1X_1X_2X_4 \end{array}\right)$$

(shows a large paper with S written out)

THE QUEEN Enough mathematics! I want to be drunk! (She drains the glass, falls to the floor, crying out in pain) Oh, OH!! (etc.)

THE KING Gertrude! What have you done!? There was poison in that wine.

(He puts a sword near Gertrude - leans over her)

THE QUEEN WHAT!? (furiously - with her last dying strength) You wanted to kill my son?! You murderer! You scoundrel! You miserable man!

(The Queen grasps the sword and plunges it into the King - blood flows from the King's heart)

THE KING (Tries to stay on his feet - he staggers) We shall never publish our solution! PUBLISH OR PERISH! OH! OH!

(The King falls to the ground, suffering. They both are dead)

ACKNOWLEDGEMENTS

Hamlet and the Queen gratefully acknowledge the continuous interest of Susanne Pumplün in this play, and its consequences. We also thank her for her correction of basis B on page 8.

Remark (July 2009). Karim Johannes Becher proved the Pfister factor conjecture in *Invent. Math.* **173** (2008), 1-6.

Susanne Pumplün was able to extend the ideas in this play to higher dimensional forms (see her forthcoming papers).

This play was performed in May 1992 at Oberwolfach with Albrecht Pfister present!